

# 2d manifold-independent spinfoam theory

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## Abstract

A number of background independent quantizations procedures have recently been employed in 4d nonperturbative quantum gravity. We investigate and illustrate these techniques and their relation in the context of a simple 2d topological theory. We discuss canonical quantization, loop or spin network states, path integral quantization over a discretization of the manifold, spin foam formulation, as well as the fully background independent definition of the theory using an auxiliary field theory on a group manifold. While several of these techniques have already been applied to this theory by Witten, the last one is novel: it allows us to give a precise meaning to the sum over topologies, and to compute background-independent and, in fact, “manifold-independent” transition amplitudes. These transition amplitudes play the role of Wightman functions of the theory. They are physical observable quantities, and the canonical structure of the theory can be reconstructed from them via a  $C^*$  algebraic GNS construction. We expect an analogous structure to be relevant in 4d quantum gravity.

## 1 Introduction

In this paper we study the quantization of the two-dimensional (2d) topological field theory characterized by the action

$$S[\omega, B] = \int \text{Tr}[BF], \quad (1)$$

where  $\omega$  is an  $SU(2)$  connection field,  $F$  its field strength two-form, and  $B$  a scalar field in the adjoint representation of  $SU(2)$ . The theory is the 2d version of BF theory [1], and can be seen as the  $e \mapsto 0$  limit ( $e$  being the coupling constant) of 2d Yang Mills theory [2]. We shall consider this theory on manifolds of arbitrary topology, without and with boundaries, as well as in a context which allows us to sum over manifold topologies.

We utilize different quantization strategies, and discuss their relation. (For a beautiful introduction to many of the ideas in this area and a detailed annotated bibliography, see [6].) First, we utilize canonical quantization, and exhibit the loop basis [4] and the spin network basis [5]. Next, we consider a path integral quantization defined in terms of a triangulation of the 2d manifold, and the spinfoam formalism [7]. We show equivalence with canonical quantization and we show that this technique

allows the theory to be defined on rather arbitrary manifolds without and with boundaries. Finally, we recover the theory from the Feynman expansion of an auxiliary field theory over a group manifold.

The two dimensional system we consider has been studied in a variety of ways, and several of the techniques used below are known. See in particular [2], the book [3], and complete references therein. The last technique, namely the auxiliary field theory over a group, on the other hand is quite recent. This technique derives from the 2d matrix model approaches to 2d quantum gravity developed in the context of “zero dimensional” string theory [8]; the idea was nontrivially extended to 3d by Boulatov [9] and to 4d by Ooguri [10, 11], then to the Barrett Crane model [12] in [13], and to rather arbitrary diffeomorphism-invariant but non-topological theories in [14]. The main interest of this technique is that it provides a natural prescription for summing over topologies. Using this technique, we can define and compute the theory’s transition amplitudes in a fully manifold-independent fashion.

These transition amplitudes capture the full physical and gauge invariant content of the theory. In this diff-invariant context, they play a role somewhat analogous to the role that the Wightman functions play for a quantum field theory over a background. As for the Wightman functions [15], one can reconstruct the canonical structure of the theory from them, using the  $C^*$  algebraic Gelfand-Naimark-Segal (GNS) construction. In addition, we show that these transition amplitudes are given by the  $n$ -point functions of the auxiliary field theory.

A technical result of this paper is the definition and the computation of these manifold-independent transition amplitudes for the theory (1). These turn out to be strictly related to the matrix models ones – the precise relation is detailed in the Appendix. The motivation for this exercise, on the other hand, is to investigate and illustrate the different quantization techniques and their relations, in view of the physical 4d theory. The auxiliary field theory technique finds its full motivation in the context of (4d) diff-invariant but non-topologically invariant theories such as general relativity. In this context, it provides the prescription for a sum over triangulations which restores the triangulation independence of the transition amplitudes [13, 14, 16]. In the simpler context studied here, triangulation independence is already guaranteed by topological invariance, and the auxiliary field theory technique is used only to define the sum over topologies. In spite of this simplification, the study of the 2d context shed considerable light on several issues emerged in the physically important 4d context, particularly on the general structure of a diffeomorphism invariant quantum field theory [17, 18]. We think that the structures and techniques developed here will play a role in the non-topological, but still diff-invariant, context of physical quantum gravity.

## 2 Classical theory

Consider the action (1) defined over a given two-dimensional manifold  $\mathcal{M}$ . Introducing coordinates  $x^a$ ,  $a = 1, 2$  on  $\mathcal{M}$  and a basis of (anti-hermitian) matrices  $\tau_i$  in  $su(2)$ , the action is written in terms of the components of the fields as

$$S = \int B^i F_{12}^i dx^1 dx^2 \quad (2)$$

(sum over repeated indices is understood.) The field equations are

$$dB = 0, \quad (3)$$

$$F = 0, \quad (4)$$

where  $d$  is the  $\omega$ -covariant exterior differential. Apparently, there are two local gauge invariances in the action: the conventional Yang-Mills-like local  $SU(2)$  transformations generated by a Lee algebra

valued scalar field  $\lambda$

$$\delta_\lambda \omega = d\lambda, \quad (5)$$

$$\delta_\lambda B = [B, \lambda], \quad (6)$$

and the active diffeomorphisms generated by a vector field  $v$

$$\delta_v \omega = L_v \omega, \quad (7)$$

$$\delta_v B = L_v B, \quad (8)$$

$L_v$  being the Lie derivative. However, on shell a diffeomorphism generated by  $v$  is the same transformation as an  $SU(2)$  transformation generated by the field

$$\lambda^i = v^a \omega_a^i, \quad (9)$$

as can be easily checked by writing these equations in components and using the equations of motion. Therefore in this theory the diffeomorphisms (acting on the space of solutions) can be considered as a subgroup of the  $SU(2)$  gauge transformations. Solutions are given by flat connections and (covariantly) constant  $B$  fields. Locally, therefore, there is no gauge-invariant degree of freedom.

### 3 Hamiltonian analysis

Assume now that  $\mathcal{M}$  has the topology  $S_1 \times R$ , and let  $x^a = (t, \phi)$ , where  $t$  is a non-compact coordinate along  $R$  and  $\phi \in [0, 2\pi]$  is a periodic coordinate on  $S_1$ . We can then write the action (2) as

$$S = \int dt \int d\phi (B^i \partial_t \omega_\phi^i - \omega_t^i D B^i), \quad (10)$$

where  $D$  is the covariant derivative of the connection  $A^i = \omega_\phi^i$  on the 1d manifold  $S_1$ . From this we read out the canonical structure of the theory: the canonical fields are  $A^i(\phi)$  and  $B^i(\phi)$ , with Poisson brackets  $\{A^i(\phi), B^j(\phi')\} = \delta^{ij} \delta(\phi, \phi')$ , and there is the single first class constraint which generates (fixed time)  $SU(2)$  gauge transformations

$$C^i = D B^i. \quad (11)$$

This confirms that the  $SU(2)$  transformations exhaust the gauge invariances of the theory. The extended phase space on which this constraint is defined is the infinite dimensional space of the initial data  $(A^i(\phi), B^i(\phi))$ . The space of the gauge orbits is finite dimensional. In fact, it is two dimensional. It can be coordinatized by the two gauge invariant quantities

$$T = \text{Tr} U[A] = \text{Tr} P e^{\oint_{S_1} A}, \quad (12)$$

$$L = B^i(0) B^i(0), \quad (13)$$

where 0 is an arbitrary point, since  $B^i(\phi) B^i(\phi)$  is constant in  $\phi$ . The two quantities  $T$  and  $L$  commute with the constraint and form a complete system of gauge invariant observables. The physical states of the theory are therefore characterized by these two quantities. All relevant information follows from this. For instance: given two set of initial data  $(A_{in}^i(\phi), B_{in}^i(\phi))$  and  $(A_{out}^i(\phi), B_{out}^i(\phi))$ , can the first evolve into the second? The answer is yes if and only if  $T[A_{in}^i, B_{in}] = T[A_{out}^i, B_{out}]$  and  $L[A_{in}^i, B_{in}] = L[A_{out}^i, B_{out}]$ , and in this case the trajectory between the two in the coordinate time  $t$  is given by any arbitrary one parameter family of gauge transformations, parametrized by  $t$ , taking the *in* fields into the *out* fields.

## 4 Canonical quantization

We begin to construct the quantum theory by quantizing the unconstrained hamiltonian theory and imposing the quantum constraint à la Dirac. The space of the unconstrained quantum states is formed by functionals of the connection  $\Psi[A]$ . The  $B$  field is represented by the functional derivative operator  $B(\phi) = -i\hbar\delta/\delta A(\phi)$ , which gives the right commutation relations. The constraint is the generator of gauge transformations on the argument of the quantum state, and therefore the states that solve the Dirac constraint are the states of the form

$$\Psi[A] = f(\text{Tr}U[A]) = \psi(U[A]). \quad (14)$$

Therefore a physical state is given by a function  $\psi$  over  $SU(2)$ , depending only on the trace of its argument – that is, invariant under the adjoint action of the group over itself:  $\psi(U) = \psi(VUV^{-1})$ . Such functions are denoted “class functions”. There is a natural invariant scalar product on these states, which is

$$\langle \Psi, \Psi' \rangle = \int dU \overline{\psi(U)} \psi'(U). \quad (15)$$

where  $dU$  is the Haar measure on  $SU(2)$ . Throughout the paper, we denote  $\mathcal{H}$  the Hilbert space of the square integrable functions over  $SU(2)$ . Then the physical Hilbert space is

$$\mathcal{H}_{ph} = \frac{\mathcal{H}}{SU(2)} = \frac{L_2[SU(2)]}{SU(2)}, \quad (16)$$

where the  $SU(2)$  in the denominator is the adjoint action of the group over itself. The gauge invariant observables  $T$  and  $L$  are well defined on  $\mathcal{H}$ . The first gives

$$T\psi(U) = \text{Tr}(U) \psi(U); \quad (17)$$

and a straightforward calculation shows that the second gives

$$L\psi(U) = \hbar^2 C\psi(U), \quad (18)$$

where  $C$  is the  $SU(2)$  Casimir operator. The essential condition on the physical scalar product on the solution of the constraint equation is that the physical operators be self-adjoint. They are, and therefore the scalar product we have defined is the physically correct one. Since the Casimir of  $SU(2)$  has eigenvalues  $j(j+1)$ , with half integer  $j$ , we obtain immediately a first result from the quantum theory: the observable  $L$  is quantized, with eigenvalues

$$L_j = \hbar^2 j(j+1). \quad (19)$$

A natural basis can be obtained by diagonalizing  $L$ . Since the Casimir operator is diagonal over the irreducible representations, we define

$$\psi_j(U) = \text{Tr}_j U \equiv \text{Tr} [R^{(j)}(U)], \quad (20)$$

where  $R^{(j)}(U)$  is the matrix representing  $U$  in the representation  $j$ . Using the well known relation

$$\int dU \overline{R_{ab}^{(j)}(U)} R_{cd}^{(k)}(U) = \frac{1}{2j+1} \delta^{jk} \delta_{ac} \delta_{bd}, \quad (21)$$

we have immediately that the  $\psi_j$  form an orthonormal basis, which we denote  $|j\rangle$ . Therefore

$$\langle U|j\rangle = Tr_j U \quad (22)$$

We have thus  $\langle j|j'\rangle = \delta_{jj'}$ . The action of  $T$  on this basis can be obtained directly from standard Clebsch Gordon technology. The normalized (generalized) eigenstates  $|T\rangle$  of the  $T$  operator form a continuous orthonormal basis, and thus we have

$$\langle T'|T\rangle = \delta(T, T'). \quad (23)$$

A useful set of operators to consider is the one corresponding to the classical observables

$$T_j = Tr_j U[A]. \quad (24)$$

These are clearly diagonal in the  $U$  representation and

$$T_j|0\rangle = |j\rangle. \quad (25)$$

Although this model is very simple, it has remarkable analogies with the loop quantization of general relativity. In loop quantum gravity as well, indeed, we have states functionals of the connection that can be written as functions of the holonomies  $U$  of the connection. While in loop quantum gravity one must consider holonomies along different loops, here there is essentially only one loop: the one that wraps around  $S_1$ . Thus, this theory can be seen as a sort of “single loop” loop quantum gravity. Indeed, (22) is an elementary loop transform. Better, the basis  $|j\rangle$  that we have introduced is precisely the “spin network basis” [5], in this simplified case of a single loop. The  $T$  operator is then analogous to the quantum gravity loop operator, and the operator  $L$  is analogous to the area operator, which is also given in terms of the Casimir of  $SU(2)$ , and which is diagonalized by the spin network basis [19]. The quantization of  $L$  is thus the 2d analog of the quantization of the area.

## 5 Discretization

Let us now consider a completely different path for quantization. We start with the covariant theory, triangulate the manifold  $\mathcal{M}$  and consider a lattice gauge theory that discretizes our theory. In general, the discretization introduced in going to the lattice kills the degrees of freedom below a certain scale; however, since the theory we are considering has no local degrees of freedom, we expect nothing to be really lost in the discretization. As we shall see, this will indeed turn out to be the case. In addition, the discretization allows us to get rid of the restriction to the simple  $S_1 \times R$  topology, and consider arbitrary topologies with an arbitrary number of boundaries.

Let us therefore fix a triangulation  $\Delta$  of  $\mathcal{M}$ . A triangulation in two dimensions is formed by triangles, edges, and points. A triangle is bounded by three edges and an edge by two (not necessarily distinct) points. An edge bounds precisely two triangles and a point bounds an arbitrary number of edges. It is more convenient to use the dual  $\Delta^*$  of a triangulation  $\Delta$ , which is formed by trivalent vertices, links and faces (or plaquettes) with an arbitrary number of sides. We discretize the connection by replacing it with a group element  $U_l$  for every link  $l$  of  $\Delta^*$ . We discretize the field  $B$  by replacing it with a Lie algebra valued variable  $B_f$  for every face  $f$ . Finally, we approximate the action (1) as a sum over the faces

$$S[B_f U_l] = \sum_f Tr[B_f U_f], \quad (26)$$

where  $U_f = U_{f_1} \dots U_{f_n}$  when  $f_1 \dots f_n$  are the links around the face  $f$ . To first order in the area of the plaquettes we have  $Tr[B_f U_f] \rightarrow Tr[B(1 + F)] = Tr[BF]$ . We then consider the quantum theory defined by the partition function

$$Z_{\mathcal{M}} = \int dB_f dU_l e^{\frac{i}{\hbar} S[B_f U_l]}. \quad (27)$$

The subscript  $\mathcal{M}$  indicates the manifold: we will indeed check shortly that  $Z_{\mathcal{M}}$  depends on the manifold but not the triangulation. The integral over  $B_f$  can be performed explicitly [20], giving

$$Z_{\mathcal{M}} = \int dU_l \prod_f \delta(U_f) = \int dU_l \prod_f \delta(U_{f_1} \dots U_{f_n}), \quad (28)$$

where  $\hbar$  and  $2\pi$  coefficients have been absorbed in a redefinition of the measure  $dB_f$ . We can then expand the delta function over  $SU(2)$ , using a well known representation of it

$$Z_{\mathcal{M}} = \int dU_l \prod_f \sum_j (2j+1) Tr_j U_f. \quad (29)$$

Equation (29) can be rearranged as

$$Z_{\mathcal{M}} = \sum_{j_f} \int dU_l \prod_f (2j_f+1) Tr_{j_f} U_f. \quad (30)$$

where  $j_f$  is an assignment of a spin  $j$  to every face  $f$ , and the sum is over all such assignments.

These two steps have replaced the integrals over the continuous variables  $B_f$  (one per face) with sums over the discrete index  $j_f$  (one per face). To understand how this may have happened, consider the following analogy. Let  $-\pi < x < \pi$  and consider a space  $S$  of sufficiently regular functions over this interval. Consider the integral

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp e^{ipx}. \quad (31)$$

It converges to a well defined distribution over  $L$ : the delta function on the origin. However, on this restricted interval the same delta function can be obtained with a discrete sum as

$$\delta(x) = \frac{1}{2\pi} \sum_n e^{inx}. \quad (32)$$

In a sense, the values  $p = n$  of  $p$  are “sufficient” for the integral. Similarly, the “quantized” values  $j$  in (30) are sufficient to give the same delta function over the group as the one defined by the  $B$  integral in (27).

The remaining integrals in (30) can be performed because each link  $l$  always bounds precisely two faces, and therefore each integration variable  $U_l$  enters in precisely two traces. Using (21), we conclude that the representation must be the same for every two adjacent faces, which is to say for all faces. Therefore the sum over  $j_f$  reduces to a single sum over  $j$ . From equation (21), each integral (that is, each link) contributes a factor  $1/(2j+1)$ , and a bunch of delta functions that end up contracted among themselves. A moment of reflection on the pattern of the indices in the integrals will convince the reader that after all the integrals have been performed, there remains one trace for each (trivalent)

vertex in  $\Delta^*$ . Each such trace gives a contribution  $(2j+1)$  to the integral. Putting all together we obtain

$$Z_{\mathcal{M}} = \sum_j \prod_f (2j+1) \prod_l 1/(2j+1) \prod_v (2j+1) = \sum_j (2j+1)^{F-L+V}, \quad (33)$$

where  $F, L, V$  are the numbers of faces links and vertices in  $\Delta^*$ . The quantity  $\chi = F - L + V$  is a topological invariant, that is, for a fixed manifold is triangulation independent. In fact, it is the Euler characteristic  $\chi(\mathcal{M})$  of  $\mathcal{M}$ . For a compact oriented surface of genus  $g$ , the Euler characteristic is  $\chi = 2 - 2g$ . Thus

$$Z_{\mathcal{M}} = \sum_j (2j+1)^{\chi(\mathcal{M})} = \sum_j (2j+1)^{2-2g(\mathcal{M})}, \quad (34)$$

which converges for  $g > 1$ , namely for all Riemann manifolds except for the sphere ( $g=0$ ) and the torus ( $g = 1$ ). For non-orientable surfaces, the Euler characteristic and the genus are linked by  $\chi = 2 - g$  and the partition function converges for  $g > 2$ , namely for all surfaces but the projective plane  $RP^2$  ( $g = 1, \chi = 1$ ) and Klein's bottle  $K$  ( $g = 2, \chi = 0$ ). As we will see in a moment, this divergence is harmless and physical quantities are all well defined.

## 6 Boundaries

The calculation in the previous section is not very meaningful by itself, since the sourceless partition function  $Z_{\mathcal{M}}$  is either to be normalized to one or infinite. We have performed it only as a preliminary step to compute something more interesting. To get to something more interesting we have to have states in the theory and transition amplitudes.

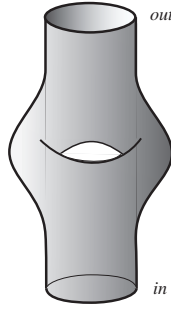


Figure 1: A spacetime manifold of genus 1 with two boundary components.

A diffeomorphism invariant field theory has the following general structure. Let the manifold  $\mathcal{M}$  have boundary  $\Sigma$ , formed by  $n$  connected components  $\Sigma_i$  with  $i = 1, \dots, n$ . Figure 1 illustrates, as an example, a manifold with genus one and a two component boundary. (Unconstrained) initial data  $A_i$  can be associated to each connected component  $\Sigma_i$  of the boundary. The path integral of the bulk variables at fixed boundary values gives the transition amplitudes  $Z_n[A_i]$ . To make the notation lighter we do not explicitly indicate the manifold dependence of these quantities, but it is important to remember that they do depend on the topology of the spacetime manifold that interpolates between the boundaries (unlikely the transition function of the next sections, which do not). In particular,  $Z_1[A]$  defines the “Hartle-Hawking” vacuum state state [21], and  $Z_2[A_{out}, A_{in}]$  defines the propagator. This propagator defines the projection operators  $P$  that projects the space of the unconstrained states

$\psi(A)$  into the physical Hilbert space,

$$P\psi(A) = \int dA' Z_2[A, A']\psi(A') \quad (35)$$

and defines the physical scalar product of the theory

$$\langle \psi | \psi' \rangle = \int dA dA' \overline{\psi(A)} Z_2[A, A'] \psi'(A'). \quad (36)$$

Therefore  $Z_1[A]$  picks a preferred state in the Hilbert space,  $Z_2[A_{out}, A_{in}]$  maps the Hilbert space associated to the *in* boundary to the Hilbert space associated to the *out* boundary,  $Z_3[A_1, A_2, A_3]$  gives a three legs transition amplitude, and so on.

The amplitudes  $Z_n[A_i]$  are related to each other as follows. Assume the two manifolds  $\mathcal{M}_1$  and  $\mathcal{M}_2$  can be glued along a common boundary  $\Sigma_j$  obtaining a manifold  $\mathcal{M}$ . The transition amplitudes of  $\mathcal{M}$  can be obtained from the ones of  $\mathcal{M}_1$  and  $\mathcal{M}_2$  by integrating on the common initial data  $A_j$  on  $\Sigma_j$ .

More formally, this relation can be expressed as follows [22]. There is a category  $M$  whose objects are collections of circles, and whose maps are 2d manifolds having these circles as boundaries. The diffeomorphism invariant field theory is then a representation of this category, that is, a functor from  $M$  to the category  $H$  of the Hilbert spaces (whose maps are linear maps between Hilbert spaces).

In the case we are considering, all boundary connected components are isomorphic to  $S_1$ , and therefore there is a single fundamental Hilbert  $\mathcal{H}$  space in the game (The Hilbert space associated to  $n$  circles is the symmetric tensor product of  $n$  copies of  $\mathcal{H}$ ). Also, using the discretized definition of the path integral, all integrals are well defined. The dual triangulation  $\Delta^*$  of  $\mathcal{M}$  induces a triangulation of each boundary component. We can always assume that this triangulation is made by a single segment, and denote the associated group element as  $U_i$ . Boundary values are therefore group elements  $U_i$ , which we can immediately identify as the holonomy of the connection around the boundary. All integration measures are given by the Haar measure. Let us compute the amplitudes  $Z_n[U_i]$ .

The Hartle-Hawking amplitude  $Z_1[U]$  of a hemisphere can be computed immediately from (28). Inserting the boundary value and discretizing the manifold with a single face, we have immediately

$$Z_1[U] = \sum_j (2j+1) \text{Tr}_j U = \delta(U). \quad (37)$$

Thus, the Hartle Hawking state  $|HH\rangle$  is the delta function on the group.

Next, consider a manifold with the topology of a cylinder. Again, we can discretize the cylinder with a single face, bounded by the two boundaries of the manifold and by an internal link joining the two boundaries. In (28), we have then a single integral and a single trace

$$\begin{aligned} Z_2[U_{out}, U_{in}] &= \int dV \delta(U_{in}^{-1} V U_{out} V^{-1}) \\ &= \sum_j (2j+1) \int dU \text{Tr}[R^{(j)}(U_{in}^{-1} U U_{out} U^{-1})]. \end{aligned} \quad (38)$$

Using (21) again, this gives

$$Z_2[U_{out}, U_{in}] = \sum_j \text{Tr}_j U_{out} \text{Tr}_j U_{in}. \quad (39)$$



This is precisely the projector on the class functions over the group

$$\begin{aligned} \int dU' Z_2[U, U'] \psi(U) &= \int dU' dV \delta(U^{-1} V U' V^{-1}) \psi'(U') \\ &= \int dV \psi'(V U V^{-1}). \end{aligned} \quad (40)$$

If  $\psi$  and  $\psi'$  are class functions

$$\langle \psi | \psi' \rangle = \int dU \overline{\psi(U)} \psi'(U). \quad (41)$$

That is, we recover (16), the same physical Hilbert space  $\mathcal{H}_{ph}$  as in the canonical theory.

In the basis  $|j\rangle$

$$Z_n(j_i) \equiv \int dU_i Z_n(U_i) \text{Tr}[R^{(j)}(U_i)], \quad (42)$$

the Hartle Hawking state is

$$Z_1(j) = \psi_{HH}(j) = \langle j | HH \rangle = 2j + 1, \quad (43)$$

and the propagator

$$Z_2(j, j') = \langle j | j' \rangle = \delta_{j j'}. \quad (44)$$

The higher transition amplitudes, as well as the transition amplitudes for manifolds with an arbitrary number of holes, can be computed in a similar fashion using the discretization of the manifold. A shortcut can be taken using the functorial properties of the transition amplitudes. For instance, let us glue a hemisphere to a cylinder, obtaining a hemisphere. Correspondingly, we should obtain a Hartle Hawking state by propagating a Hartle Hawking state  $\psi_{HH}(j) = \sum_j Z_2(j, j') \psi_{HH}(j)$ . In coordinate space

$$\int dU' Z_2(U, U') Z_1(U') = Z_1(U); \quad (45)$$

explicitly

$$\int dV dU' \delta(U V U' V^{-1}) \delta(U') = \delta(U). \quad (46)$$

Let us now cut out a disk from a cylinder. We obtain a manifold with three boundaries. The corresponding amplitude  $Z_3(j, j', j'')$  must satisfy

$$Z_2(j, j') = \sum_{j''} Z_3(j, j', j'') Z_1(j''), \quad (47)$$

and we have immediately

$$Z_3(j, j', j'') = \frac{1}{2j + 1} \delta_{j, j', j''} \quad (48)$$

where  $\delta_{j, j', j''}$  is one if  $j = j' = j''$ , and zero otherwise. And, similarly, we obtain

$$Z_n(j_i) = \frac{1}{(2j + 1)^{n-2}} \delta_{j_i} \quad (49)$$

where  $\delta_{j_i}$  is one if all  $j_i$  are the same and zero otherwise. In term of boundary group elements, one gets:

$$Z_n(U_i) = \int dV_i \delta \left( \prod_{i=1}^n V_i^{-1} U_i V_i \right) = \int dV_2 \dots dV_n \delta(U_1 V_2^{-1} U_2 V_2 \dots V_n^{-1} U_n V_n) \quad (50)$$

, that is the delta function on the product of the conjugacy classes of the boundary holonomies.

Then we can deduce the higher genus case by gluing some punctures together. For example, considering  $Z_3$  and identifying two punctures together, one gets the partition function for the punctured torus  $Z_1^{(g=1)}$ , and so on.

This way, we get the partition function for an orientable (compact) surface of genus  $g$

$$Z^{(g)} = \int \prod_{a=1}^g dC_a dD_a \delta \left( \prod_i C_a D_a C_a^{-1} D_a^{-1} \right), \quad (51)$$

where the group elements  $C_i, D_i$  are associated to the  $2g$  non-trivial loops/cycles of the surface. Then one immediately recovers equation (34)

$$Z^{(g)} = \sum_j (2j+1)^{2-2g}.$$

Let us point out that taking the cylinder or two-punctured sphere, we can get both the torus or Klein's bottle by gluing the two ends together, which implies that the two surfaces have the same partition function. Indeed they have the Euler characteristic  $\chi = 0$ .

Similarly, we can start with a manifold of genus  $g$  and cut out a disk. By the gluing technique, we immediately get:

$$Z_1^{(g)}(U) = \int \prod_{a=1}^g dC_a dD_a \int dV \delta \left( V U V^{-1} \prod_a C_a D_a C_a^{-1} D_a^{-1} \right), \quad (52)$$

which leads to

$$Z_1^{(g)}(j) = (2j+1)^{1-2g}, \quad (53)$$

And we finally get easily the general expression for the amplitude of a genus  $g$  manifold with  $n$  disk removed, and therefore  $n$  boundary components

$$Z_n^{(g)}(U_i) = \int \prod_{a=1}^g dC_a dD_a \int \prod_{i=1}^n dV_i \delta \left( \prod_i V_i U_i V_i^{-1} \prod_a C_a D_a C_a^{-1} D_a^{-1} \right) \quad (54)$$

$$Z_n^{(g)}(j_i) = (2j+1)^{2-2g-n} \delta_{j_i} \quad (55)$$

Notice that all these transition functions are finite.

The above discussion illustrates in detail the relation with canonical quantization. In particular, notice that the propagator associated to the cylinder is precisely the projector on the solutions of the quantum constraint. Notice that this is precisely the structure in the formal functional quantization of general relativity in Hawking's approach. Notice also that, as it is always the case in diffeomorphism invariant theories,

- the matrix elements of the physical scalar product,
  - the projector on the physical states and
  - the evolution operator in coordinate time
- are all identified.

The difference between a topological field theory and a field theory which is diffeomorphism invariant, but is not topological, is only in the number of the degrees of freedom involved. In other words,

the general structure that we expect is the same, but the Hilbert space associated to a boundary has a much richer structure. Furthermore, if we define the theory by means of a discretization, we do not expect triangulation independence, because a fixed triangulation cuts off the number of degrees of freedom.

The quantization we have discussed reproduces, in simplified form, the state-sum definition of  $BF$  theory in 3d and 4d, as in the models developed by Ponzano and Regge and by Turaev and Viro, and by Turaev, Ooguri, Crane and Yetter [23]. The analog for general relativity is given by the state sum formulations of Barret and Crane [12] and their variants [16], where, however, one loses triangulation invariance. Notice in particular the sum over assignments of representations to faces in equation (30). This sum corresponds to the state sum of these models. Here, of course, the ‘‘Clebsch Gordon’’ condition on the links are simplified by the fact that only two faces join at a link and therefore the representation of the two faces must be same, thus collapsing the sum over arbitrary colorings to a sum over a single  $j$ .

## 7 Auxiliary field theory

We now come to a main part of the paper. Following the ideas in [9, 10, 13, 14] we define an auxiliary field theory on a group manifold, whose Feynman graph expansion gives the above 2d  $BF$  theory.

Let  $g_i$ , for  $i = 1, 2$  be in  $SU(2)$ ; we change notation for the group elements for consistency with the conventions in this area and also to emphasize the different role that group elements assume now. Consider a real scalar field  $\Phi(g_1, g_2)$  on  $SU(2) \times SU(2)$ , having the following two properties. Symmetry

$$\Phi(g_1, g_2) = \Phi(g_2, g_1), \quad (56)$$

and right  $SU(2)$  invariance

$$\Phi(g_1, g_2) = \Phi(g_1 g, g_2 g), \quad \forall g \in SU(2). \quad (57)$$

These two symmetries can also be expressed by writing the field as

$$\Phi(g_1, g_2) = \int_{SU(2)} dg \left( \Psi(g_1 g, g_2 g) + \Psi(g_2 g, g_1 g) \right), \quad (58)$$

where  $\Psi(g_1, g_2)$  is an arbitrary field on  $SU(2) \times SU(2)$ . The field theory is defined by the nonlocal action

$$\begin{aligned} S &= \int_{SU(2) \times SU(2)} dg_1 dg_2 \Phi^2(g_1, g_2) \\ &+ \frac{\lambda}{3!} \int_{SU(2) \times SU(2) \times SU(2)} dg_1 dg_2 dg_3 \Phi(g_1, g_2) \Phi(g_2, g_3) \Phi(g_3, g_1). \end{aligned} \quad (59)$$

Notice that in each of the two terms each integration variable is shared by precisely two fields. The structure of the two terms, ‘‘kinetic’’ and ‘‘potential’’, can be represented as in Figure 2, where the circles represent the fields and the lines represent their shared arguments which are integrated over.

We now study the perturbative Feynman expansion of the theory. Because of the symmetries of the field, care should be taken in inverting the kinetic term on the subspace of the symmetric fields

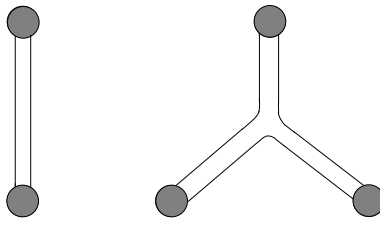


Figure 2: Structure of kinetic and potential term in the action.

only. On this subspace the kinetic term is indeed diagonal, and it is therefore straightforward to see that the propagator  $P$  and the vertex  $V$  can be written as

$$P(g_1, g_2; g'_1, g'_2) = \int dg (\delta(g_1 g, g'_1) \delta(g_2 g, g'_2) + \delta(g_1 g, g'_2) \delta(g_2 g, g'_1)) \quad (60)$$

and

$$V(g_1, g_2; g'_1, g'_2; g''_1, g''_2) = \delta(g_1, g'_2) \delta(g'_1, g''_2) \delta(g''_1, g_2) \quad (61)$$

By representing a delta function with a line, with the two end points representing the two arguments, and the group integration as two dots, we can represent the propagator and the vertex as in Figure 3.

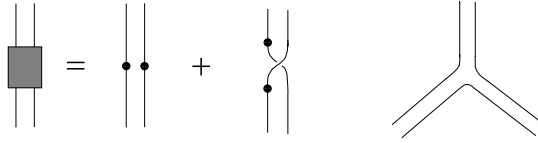


Figure 3: Propagator and vertex.

Consider the Feynman graph expansion of the the partition function

$$Z = e^F = \int d\Phi e^{-S[\Phi]}, \quad (62)$$

The partition function  $Z$  is the sum of the amplitudes of all closed Feynman graphs. The “Free energy”  $F$  is the given by the connected graphs. Consider a connected Feynman graph of order  $V$ , that is with  $V$  vertices. It will have  $E = \frac{3}{2}V$  propagators. Because of the sum in (60), which symmetrizes the arguments of the two deltas, the amplitude of the graph is the sum of  $2^E$  terms. Each of these terms can be represented by replacing the propagator in Figure 3 with one or the other of the terms in the sum, in each of the  $E$  propagators. Consider one of these terms, as for example in Figure 4. Notice that the corresponding graphical representation is formed by a graph on which closed lines run along propagators and through vertices. We denote “face” one of these closed lines, and say that it is bounded by the “edges” formed by the propagators along which the line runs. The collection  $\Gamma$  of faces, links and vertices with their boundary relations, defines a two-complex, characterized by the fact that each vertex is three-valent (it bounds three edges) and each edge is bi-valent (it bounds two faces). We denote a two complex with trivalent vertices and bivalent edges as a 2d two-complex. The Feynman amplitude of the 2d two-complex  $\Gamma$  is just given by a multiple integral of  $2E + 3V$  delta functions. Each integration variable enters in two delta functions. The pattern in which these are enchainned is

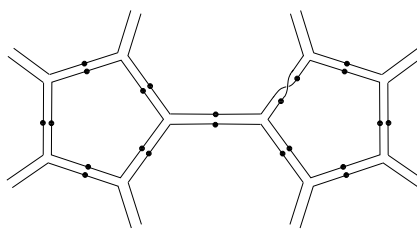


Figure 4: A 2d two-complex (part of it).

simply given by the closed lines around the faces. In other words, there is a closed sequence of deltas for each face. By integrating away all the variables at the end points of the propagators, we have then easily that the amplitude of the two-complex is

$$A_\Gamma = \int dg_e \prod_f \delta(g_{f1} \dots g_{fn}) \quad (63)$$

where  $f1, \dots, fn$  are the edges that bound the face  $f$ .

Now, consider a 2d manifold  $\mathcal{M}$  with a triangulation  $\Delta$ . Notice that the dual triangulation  $\Delta^*$  is precisely a 2d two-complex  $\Gamma$ . Furthermore, note that if  $\Gamma$  is the dual of a triangulation of  $\mathcal{M}$ , then the amplitude (63) is *precisely* the partition function of BF theory on  $\mathcal{M}$ , given above in equation (28). That is

$$Z_{\mathcal{M}} = A_{\Delta^*} \quad (64)$$

if  $\Delta$  is a triangulation of  $\mathcal{M}$ . This is a key result.

The dual of a triangulation of a two-manifold is a 2d two complex. In two dimensions, the converse is true as well. Namely each 2d two-complex is the dual of the triangulation of a 2d manifold [24]. The same is not true in higher dimensions, where one is lead to consider arbitrary two-complexes as generalized manifolds, but such complications are absent in 2d.

Just to get a feeling of the relation between Feynman graphs and 2d manifolds, consider the example of a simple graph formed by two vertices connected by three propagators. In expanding the symmetrization in the propagators, we obtain  $2^3$  terms. Let us analyze first the one in which none of the lines crosses in a planar representation, as in Figure 5a. It is immediate to see that this is the dual of a simple triangulation of a sphere obtained by gluing two triangles along their perimeter. Indeed, the two-complex has three faces, and the triangulation has therefore three (bivalent) vertices. The Euler number is  $\chi(\Gamma) = F - E + V = 3 - 3 + 2 = 2$  and the genus is zero. Next, consider the term in which there is a crossing in each propagator, as in Figure 5b. Following the line, we see that this two complex has a simple face: the corresponding triangulation has a single vertex, and a moment of reflection shows that it triangulates a torus. Indeed, the Euler number is  $\chi(\Gamma) = 1 - 3 + 2 = 0$  and the genus is one. Finally, consider the case in which only one propagator has a crossing (Figure 5c). In this case there are two faces. The corresponding manifold is  $RP^2$ , the non orientable manifold obtained from a disk by identifying opposite points on the perimeter. The Euler number is  $\chi(\Gamma) = 2 - 3 + 2 = 1$ .

We can then immediately perform the integrals in (63) as we did in the previous sections, obtaining

$$A_\Gamma = \lambda^V (2j + 1)^{\chi(\Gamma)} \quad (65)$$

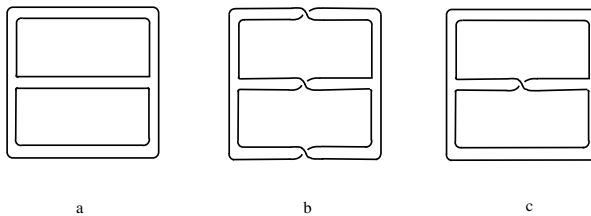


Figure 5: Two-complexes corresponding in 2d to the sphere  $S$ , the torus  $T$ , and the projective plane  $RP^2$

Now, let  $N(V, \chi)$  be the number of 2d two-complexes with Euler number  $\chi$  obtained by expanding all graphs with  $V$  vertices. This number is clearly finite and its determination is, in principle, a well defined combinatorial problem. The sum  $Z$  over the connected Feynman diagrams gives

$$Z = \sum_{\chi} N_{\chi} \sum_j (2j + 1)^{\chi}, \quad (66)$$

where the weight factor  $N_{\chi}$  is given by

$$N_{\chi} = \sum_V N(V, \chi) \lambda^V. \quad (67)$$

This defines a version of the topological field theory in which a sum over the manifold topologies is naturally implemented. The resulting theory does not depend on any fixed underlying manifold structure. The different spacetime manifold topologies are generated as Feynman graphs of the auxiliary theory. The auxiliary field theory fixes a well defined prescription for the sum over these topologies. The possibility of transitions through disconnected spacetime manifolds (for instance, two circles going into two circles via two cylinders) is taken into account by the standard field combinatorics of the connected/disconnected (reducible/irreducible) graphs. In the appendix, we discuss the precise relation between this theory and the matrix models. As shown in the appendix, the transition amplitudes between states of color  $j$  are essentially the transition amplitudes of a matrix model with  $N = 2j + 1$  dimensional matrices [3, 8, 11]. For the discussion of their finiteness, see for instance [3, 11].

## 8 Transition amplitudes

Once more, the physical content of the theory is not in its sourceless partition function but in its transition amplitudes, which can be obtained by inserting field operators in the path integral, coupling a source and taking derivatives with respect to the source, or computing the partition functions with fixed boundary conditions on the fields on spacetime boundaries. Remarkably, the transition amplitudes of the topological theory, obtained by defining the theory on a manifold with boundaries, are directly connected to the  $n$ -point functions of the auxiliary theory, obtained by inserting fields in the path integral.

The easiest way to see this, is to fix the gauge in the auxiliary field. Notice indeed that because of the invariance (58), we can consider the quantity

$$\psi(g) = \Phi(1, g), \quad (68)$$

or, equivalently

$$\psi(g_2 g_1^{-1}) = \Phi(g_1, g_2), \quad (69)$$

which has the same information as the field. We thus define the  $n$ -point functions of the auxiliary field theory as

$$W(g^{(1)}, \dots, g^{(n)}) = Z^{-1} \int [D\Phi] \psi(g^{(1)}) \dots \psi(g^{(n)}) e^{-S[\Phi]}. \quad (70)$$

The momentum space version of the transition functions are

$$W_{j_1 \dots j_n} = Z^{-1} \int [D\Phi] \Phi_{j_1} \dots \Phi_{j_n} e^{-S[\Phi]}, \quad (71)$$

where

$$\Phi_j = \int dg_1 dg_2 \text{Tr}_j(g_2 g_1^{-1}) \Phi(g_1, g_2). \quad (72)$$

One could consider the more general observables

$$\Phi_j^{(\alpha)} = \int dg_1 dg_2 \text{Tr}_j((g_2 g_1^{-1})^\alpha) \Phi(g_1, g_2), \quad (73)$$

however  $\text{Tr}_j(g^\alpha)$  is simply a linear combinaison of traces  $\text{Tr}_k(g)$  with  $k$  ranging from 0 to  $\alpha j$  (due to the decomposition into irreducible representations of the representation  $j \otimes j \otimes \dots \otimes j$ ).

Consider an irreducible Feynman graph in the perturbative expansion of  $W(g^{(1)}, \dots, g^{(n)})$ . A moment of reflection, will convince the reader that this is given by the same graphs as before, with the difference that now  $n$  propagator lines are open. The group element  $g^i$  is associated to the  $i$ -th open propagator line, and in expanding the enchainment of the deltas, the two individual lines of the propagator get connected to each other through  $g^i$ . This can be represented by a circle formed by a *single* line, colored with  $g^i$ . Now, this is precisely the representation of a dual triangulation of a manifold with  $n$  boundaries, colored with group elements  $g^1, \dots, g^n$ . Therefore the Feynman expansion of the  $n$ -points function is the sum of the transition amplitudes of the topological theory, computed on manifolds with  $n$  boundaries, and summed over all topologies of the interpolating spacetime manifold.

Going to momentum space, we have then immediately, for the irreducible  $n$ -point function  $\Gamma_{j_1 \dots j_n}$

$$\Gamma_{j_1 \dots j_n} = \sum_{\chi} N_{\chi, n} (2j + 1)^{\chi - n} \delta(j_i) \quad (74)$$

where

$$N_{\chi, n} = \sum_m N(m, \chi, n) \lambda^m, \quad (75)$$

where  $N(m, \chi, n)$  is the number of irreducible (that is, connected) 2d two-complexes with  $n$  boundaries,  $m$  vertices and Euler number  $\chi$ . The relation between the transition amplitudes and their irreducible part can be obtained by standard methods: we define the generating functional of the connected graphs as a function of a source class function  $J(g)$  with components  $J_j$ .

$$\Gamma[J] = \sum_n \sum_{j_1 \dots j_n} \Gamma_{j_1 \dots j_n} J^{j_1} \dots J^{j_n} \quad (76)$$

and the generating functional of the transition amplitudes

$$W[J] = \sum_n \sum_{j_1 \dots j_n} W_{j_1 \dots j_n} J^{j_1} \dots J^{j_n}. \quad (77)$$

And we have

$$W[J] = e^{\Gamma[J]} = Z^{-1} \int d\Phi e^{-S[\Phi] + J \cdot \Phi}, \quad (78)$$

where

$$J \cdot \Phi = \sum_j \Phi_j J^j = \int dg_1 dg_2 \Phi(g_1, g_2) J(g_2 g_1^{-1}). \quad (79)$$

Notice that the divergence due to the sphere and the torus in  $Z$  does not affect the transition functions, since it is always divided out by the  $Z^{-1}$  factor in (70); equivalently, closed disconnected Feynman diagrams do not contribute to the transition amplitudes.

We have thus obtained the explicit form of all the transition amplitudes, in a form that is independent from the topology of the underlying spacetime manifold. Rather, these can be viewed as the transition amplitudes computed by summing over all topologies of the interpolating spacetime manifold. The prescription for the relative weights is implicitly fixed by the auxiliary field theory. These transition amplitudes can be directly computed as the  $n$ -point functions of the auxiliary field theory. The sum defining transition amplitudes can be viewed as a sum over the colored 2d two-complexes.

A  $n$ -dimensional analog of (59) is obtained by taking a field which is a function of  $n$  group elements, and a potential term of order  $\frac{1}{2}n(n-1)$  having the structure (see Figure 2) of an  $n$  dimensional simplex. The expansion of the Feynman graphs generates then  $nd$  two-complexes. The dual of an  $n$  dimensional triangulation is an  $nd$  two-complex. This construction provides a manifold independent definition on  $BF$  theory in  $n$  dimensions.

What is particularly remarkable is that the constraint that reduced 4d  $BF$  theory to general relativity can be obtained in the auxiliary field theory simply by requiring an additional invariance of the field under a subgroup for the field over the group [13, 16]. The 4d sum is therefore over colored 4d two-complexes. These are complexes in which each edge bounds four faces and each vertex bounds ten edges. The representations associated to the faces can vary freely provided that Clebsch-Gordon conditions are respected at the edges. Edges carry the additional degree of freedom given by the intertwiner between the representations associated to the adjacent faces. Such colored two complexes are denoted spinfoams and transition amplitudes can therefore be expressed as sums over spinfoams, hence the denomination spinfoam models.

The remarkable aspect of this formalism, when applied to gravitational theories, is that a spinfoam admits an interpretation as a discretized 4-geometry, and therefore the sum over spin foams turns out to be a well defined version of the Misner-Hawking sum of geometries [25]. In particular, it turns out that one of the Casimirs of the representation associated to the face represents the area of the face, thus giving a metric interpretation to the coloring. This can happen because, as we have seen the Casimir is quadratic in the  $B$  field. The constraint that reduces  $BF$  theory to general relativity forces the  $B$  field to be the product of two tetrad fields, and thus the Casimir is a product of four tetrads associated to a face, and a straightforward calculation shows that it is the square of the area of the face. Furthermore, the boundary Hilbert space turns out to be precisely the Hilbert space of loop quantum gravity, thus providing a precise link with the canonical formalism. The hope is that the auxiliary field theory technique could allow us to define and compute 3-geometry to 3-geometry transition amplitudes as sums over spinfoams, where the sum takes into account the full sum over topologies. As we have seen this hope is realized at least in the very simple context of the 2d topological theory.



## 9 Canonical and algebraic structure

Since the spacetime boundary can be composed by an arbitrary number of circles, a generic state of the system is the symmetric tensor product of an arbitrary number of copies of  $\mathcal{H}$ . That is, it is the Fock space  $\mathcal{F}$  over  $\mathcal{H}$ .

$$\mathcal{F} = \bigoplus_{n=0,\infty} (\mathcal{H}_1 \otimes_s \dots \otimes_s \mathcal{H}_n), \quad (80)$$

where all  $\mathcal{H}_i$ 's are isomorphic to  $\mathcal{H}$ . A basis in  $\mathcal{F}$  is given by the vectors  $|j_1 \dots j_n\rangle$ , where  $n$  is arbitrary and the set is ordered. The vacuum state  $|O\rangle$  (not to be confused with the  $|j=0\rangle$  state in  $\mathcal{H}$ ), that is, a normalized vector of the  $n=0$  term in (80), represents the absence of any boundary.

What are the physical operators on this Hilbert space and to which physical observables do they correspond? Recall that we identified only two observables in the canonical analysis of the classical theory,  $T$  and  $L$ . However, that result was under the assumption that the physical state is defined on a boundary formed by a single component. If the boundary has  $n$  components, there must be  $n$  different values  $T_1, \dots, T_n$  and  $L_1, \dots, L_n$  of these observables, representing the trace of the holonomy of the connection and the length of the (constant)  $B$  field in each of the components. In other words, the classical phase space becomes the disjoint collection of an infinite set of components, labelled by the number of circles  $n$ . In each of these, the observables can be taken to be  $T_1, \dots, T_n$  and  $L_1, \dots, L_n$ . It is also convenient to define an observable  $N$  that takes the value  $n$  on the  $n$ -th phase space component. The operator  $N$  and all the  $T_i$ 's are simultaneously diagonalized, by the basis  $|j_1 \dots j_n\rangle$ .

We now introduce the additional operator  $T_j$ , which increases  $n$  by one:

$$T_j |j_1 \dots j_n\rangle = |j_1 \dots j_n, j\rangle. \quad (81)$$

Intuitively, this operator can be understood as follows. When measuring the holonomy on a boundary with  $n$  circles, we have to specify on which circle we are measuring it. There are therefore  $n$  distinct holonomy operators acting on  $|j_1 \dots j_n\rangle$ , each acting on a different  $j_i$ . But this is not sufficient, since we can also measure the holonomy of a next extra circle opening up, and  $T_j$  is related to this operation.

Observable quantities of the theory are the transition functions  $W_{j_1 \dots j_n}$ . These have a straightforward physical interpretation as follows. We can arbitrarily divide the  $n$  indices  $j_1, \dots, j_n$  into two families: the *in* ones  $|j_1^{in} \dots j_{n_{in}}^{in}\rangle$  and the *out* ones  $|j_1^{out} \dots j_{n_{out}}^{out}\rangle$ . Then  $W_{j_1 \dots j_n}$  is the probability amplitude to find the system in the  $|j_1^{out} \dots j_{n_{out}}^{out}\rangle$  state after having found it in the  $|j_1^{in} \dots j_{n_{in}}^{in}\rangle$ .

Notice that we can write

$$|j_1 \dots j_n\rangle = T_{j_1} \dots T_{j_n} |O\rangle \quad (82)$$

and

$$W_{j_1 \dots j_n} = \langle 0 | T_{j_1} \dots T_{j_n} | O \rangle \quad (83)$$

The  $W_{j_1 \dots j_n}$  functions, and their coordinate space transform  $W(g^{(1)}, \dots, g^{(n)})$ , are therefore the vacuum expectation values of products of  $T_j$  operators.

Conversely, assume that the  $W(g^{(1)}, \dots, g^{(n)})$ , functions are given to us. Then we can reconstruct the quantum theory from them, in the spirit of Wightman. To do that, consider a linear  $\mathcal{L}$  space of sufficiently regular “test” functions  $f(g^{(1)}, \dots, g^{(n)})$ . We can promote  $\mathcal{L}$  to a  $C^*$  algebra by defining the adjoint  $f^*(g^{(1)}, \dots, g^{(n)}) = \overline{f(g^{(n)}, \dots, g^{(1)})}$ , the norm  $|f| = \sup[f(g^{(1)}, \dots, g^{(n)})]$ , the product  $(fh)(g^{(1)}, \dots, g^{(n+m)}) = f(g^{(1)}, \dots, g^{(n)}) h(g^{(n+1)}, \dots, g^{(n+m)})$ . The  $W(g^{(1)}, \dots, g^{(n)})$  functions define a positive linear functional  $W$  on  $\mathcal{L}$  by

$$W(f) = \sum_n \int dg^{(1)} \dots dg^{(n)} W(g^{(1)}, \dots, g^{(n)}) f(g^{(1)}, \dots, g^{(n)}) \quad (84)$$

We can thus run the GNS construction and obtain an Hilbert space, an algebra of operators and a vacuum state such that  $W$  is the vacuum expectation value of the operators in the algebra. We can reconstruct in this way the hamiltonian structure defined above. Therefore the hamiltonian structure of the theory can be reconstructed from the transition amplitudes, in the spirit of Wightman.

Notice that the pre-Hilbert scalar product  $(f, h) = W(f^*h)$  that the GNS construction is degenerate, because of the  $g \mapsto g'gg'^{-1}$  invariance under conjugation of the two point function and therefore  $\mathcal{L}$  is projected to  $\mathcal{L}_{ph}$ , which is formed by class functions only. For instance, the function  $f(g) = f^{ab}R_{ab}(g)$ , with  $Tr f = 0$ , has zero norm and is projected out of  $L_{ph}$  in the GNS construction of the Hilbert space.  $\mathcal{L}_{ph}$  is then spanned by the basis  $f^{j_1 \dots j_n}$  of the components of the class functions  $f$ , and

$$W(f) = W_{j_1 \dots j_n} f^{j_1 \dots j_n}. \quad (85)$$

In conclusion, the  $W_{j_1 \dots j_n}$  functions are gauge invariant, capture the full content of the theory and correspond to physical observables with a clear interpretation: they are therefore a natural set of objects in terms of which to deal with the theory and its physical content. Analogous objects for 4d quantum gravity exist and will be described elsewhere.

## 10 Conclusions and perspectives

We have studied the quantization of a simple diffeomorphism invariant theory using several techniques, which turn out to be nicely consistent. The auxiliary field theory method provides a prescription for a manifold-independent definitions of the theory, and a way to compute manifold-independent transition amplitudes. These can be understood as sums over different spacetime topologies. They capture the physical content of the theory, and represent physical observable quantities [26].

The interest of this auxiliary field theory technique is that it extends to quantum gravity. Indeed, there is an auxiliary field theory formulations of quantum gravity [13, 14, 16]. In this case, the Feynman graph sum provides a genuine sum over triangulations, which erases the triangulation dependence of the non-topological theory. Furthermore, the Hilbert space  $\mathcal{H}_{ph}$  associated to a boundary component turns out to be precisely the Hilbert space of loop quantum gravity. In particular, the  $W$  functions of quantum gravity capture its full diff-invariant physical content, and we expect that they could be obtained as the  $n$ -point functions of an auxiliary field theory such as the one defined in [16]. Thus, a beautifully overall covariant and canonical coherent picture of nonperturbative quantum gravity seems to be emerging.

A word of caution should be added, regarding the general use of the auxiliary field theory and the possibility of computing transition amplitudes in this path integral form. As we have seen, the two point function is the projector on the solutions of the canonical constraints. The projector should have zero eigenvalues. These correspond to the unphysical gauge degrees of freedom, which should be projected out. As well known, gauge degrees of freedom may give divergences in the path integral, essentially due to the integration over the infinite volume of the gauge group. These may be cured; in the theory in this paper, care has been taken in inverting the propagator on the physical subspace only, thus getting rid of an infinity. In a more complicated case, getting rid of those infinities might not be as simple and zeros of the projector might appear as divergences. This issue might be harder to deal with if the potential term alters the invariances of the kinetic term.

Finally, it would be important to study the auxiliary field theory formulation and the algebraic structure in the case of an arbitrary group in which irreducible representations are not conjugate to

themselves. This might affect the past/future structure of the transition amplitudes, and the star relation in the  $C^*$  algebra.

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## Appendix: Relation with the matrix models

Let us expand the field in mode components, using the Peter-Weyl theorem, as follows

$$\Phi(g_1, g_2) = \phi_{j_1 j_2}^{a_1 b_1 a_2 b_2} \overline{R_{a_1 b_1}^{j_1}(g_1)} R_{a_2 b_2}^{j_2}(g_2). \quad (86)$$

The symmetry property (57) of the field implies

$$\Phi(g_1, g_2) = \int dg \phi(g_1 g, g_2 g) \quad (87)$$

$$= \phi_{j_1 j_2}^{a_1 b_1 a_2 b_2} \overline{R_{a_1 c_1}^{j_1}(g_1)} R_{a_2 c_2}^{j_2}(g_2) \int dg \overline{R_{c_1 b_1}^{j_1}(g)} R_{c_2 b_2}^{j_2}(g). \quad (88)$$

Using (21), we can write

$$\Phi(g_1, g_2) = \sqrt{2j+1} \phi_j^{a_1 a_2} \overline{R_{a_1 c}^j(g)} R_{a_2 c}^j(g) \quad (89)$$

where we have defined

$$\phi_j^{a_1 a_2} = \frac{1}{\sqrt{2j+1}} \phi_{j j_2}^{a_1 b_1 a_2 b_2} \delta_{b_1 b_2} \delta^{j j_2}. \quad (90)$$

The symmetry property (56) implies  $\phi_j^{a_1 a_2} = \overline{\phi_j^{a_2 a_1}}$ . Writing the action (59) in terms of these modes, we obtain for the kinetic term

$$\int \Phi^2(g_1, g_2) dg_1 dg_2 = \phi_j^{a_1 a_2} \phi_j^{a_1 a_2}, \quad (91)$$

and for the potential term

$$\frac{\lambda}{3!} \int \phi(g_1, g_2) \phi(g_2, g_3) \phi(g_3, g_1) dg_1 dg_2 dg_3 = \frac{\lambda}{3!} \frac{1}{\sqrt{2j+1}} \phi_j^{ab} \phi_j^{bc} \phi_j^{ca} \quad (92)$$

The action is a sum over  $j$  of terms  $S_j$ . For each of these terms, we define an hermitian matrix  $M$ , of dimension  $N = 2j+1$  by  $M_{ab} = \phi_j^{ab}$ . Then each term takes the form

$$S_j = \frac{1}{2} \text{Tr}(M^2) + \frac{\lambda}{3!} \frac{1}{\sqrt{N}} \text{Tr}(M^3) \quad (93)$$

which is a standard form for the matrix models action [8, 10]. Therefore our theory is formed by a collection of non interacting matrix models, one per representation.

When calculating the Feynman diagrams from the above model, the propagators (corresponding to the edges of the dual triangulations) don't give any weight, the vertices give each a weight  $N^{-1/2}$

and the loops in the diagrams (corresponding to faces of the dual triangulation) give the weight  $N$  (size of the matrices), so that the total weight of a diagrams is

$$\mathcal{W} = \lambda^V N^{F - \frac{1}{2}V} \quad (94)$$

Using  $2E = 3V$ , we find that the exponent is  $F - \frac{1}{2}V = F - E + V$ , the Euler characteristic of the manifold triangulated by the (dual of the) Feynman diagram. Notice also that, if one scales the variables  $\phi_j^{ab}$  or  $M_a^b$  by a factor  $\alpha$ , the propagators gets a factor  $\alpha^{-2}$  and the vertices a factor  $\alpha^3$ , so the overall factor will be  $\alpha^{3V-2E} = 1$  and does not affect the total weight.

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